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Local uniqueness of generalized Shalika models for SO_{4n}

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ABSTRACT

This paper shows the uniqueness of generalized Shalika model on $SO(4n, F)$, $n \geqslant 1$, where F is a p-adic field or a finite field with characteristic p, $p \neq 2$.

1. Introduction

Denote by $\mathbb{F}_q$ a finite field and by $\mathcal{F}$ a p-adic field. Denote by F, either $\mathcal{F}$ or a finite field $\mathbb{F}_q$ with characteristic p, $p \neq 2$. Groups will always be considered over F unless states otherwise. When we discuss the case of p-adic groups, representations will always refer to as **smooth representations** without emphasizing the word “smooth”.

Denote by $\mathrm{Mat}_{m\times r}$ the set of $m \times r$ matrices and let Mat_m be the set of square matrices of size m. Let $\mathrm{P}_{n,n} = \mathrm{M}_{n,n}\mathrm{N}_{n,n}$ be the maximal parabolic subgroup of GL_{2n}, with

$$\mathrm{M}_{n,n} = \mathrm{GL}_n \times \mathrm{GL}_n,$$

and

$$\mathrm{N}_{n,n} := \left\{ n(X) = \begin{pmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n \end{pmatrix} \in \mathrm{GL}_{2n} \right\}.$$

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Let ψ be a fixed nontrivial character of $\mathcal{F}$. Let

$$\mathrm{GL}_n^{\Delta} = \{ d(g) = \mathrm{diag}(g, g) \mid g \in \mathrm{GL}_n \}$$

be the diagonal embedding of GL_n into $\mathrm{M}_{n,n}$. Denote by

$$\mathcal{S}_n = \mathrm{GL}_n^{\Delta} \rtimes \mathrm{N}_{n,n} \tag{1.1}$$

the **Shalika subgroup**. The **Shalika character** is given by

$$\psi_{\mathcal{S}_n}\big(d(g)n(X)\big) = \psi\big(\mathrm{tr}(X)\big).$$

Let ρ be an irreducible representation of GL_{2n}.

Definition 1.1. A linear functional $\Lambda_\rho : V_\rho \mapsto \mathbb{C}$ is called a **Shalika functional** of V_ρ if it satisfies

$$\Lambda_\rho\big(\rho(s)v\big) = \psi_{\mathcal{S}_n}(s)\Lambda_\rho(v) \quad \text{for all } s \in \mathcal{S}_n \text{ and } v \in V_\rho. \tag{1.2}$$

We say that V_ρ has a **Shalika model** if there exists a nontrivial Shalika functional Λ_ρ satisfying the above equation. This definition is equivalent to

$$\dim \mathrm{Hom}_{\mathrm{GL}_{2n}}\big(\rho, \mathrm{Ind}_{\mathcal{S}_n}^{\mathrm{GL}_{2n}} \psi_{\mathcal{S}_n}\big) \geqslant 1,$$

since $\mathrm{Hom}_{\mathrm{GL}_{2n}}(\rho, \mathrm{Ind}_{\mathcal{S}_n}^{\mathrm{GL}_{2n}} \psi_{\mathcal{S}_n}) \cong \mathrm{Hom}_{\mathcal{S}_n}(\rho|_{\mathcal{S}_n}, \psi_{\mathcal{S}_n})$ by reciprocity.

Uniqueness of Shalika models for GL_{2n} over p-adic fields was first proved in 'Uniqueness of linear periods' [JR96], by H. Jacquet and S. Rallis via the verification of multiplicity freeness of linear models and the fact that the existence of Shalika models of GL_{2n} implies existence of linear models. The result is re-obtained by the author in [N07], where cases over finite fields and p-adic quaternion division algebras are treated simultaneously. Hence the dimension of the space $\mathrm{Hom}_{\mathcal{S}_n}(V_\tau, \psi_{\mathcal{S}_n})$ is at most one for any irreducible representation V_τ.

Write $g^{-\rho} = (g^\rho)^{-1} = (g^{-1})^\rho$, $g \in \mathrm{Mat}_n$ for any operator ρ on Mat_n, which commutes with the inverse operator. Let g^t be the transpose of a matrix g. Let

$$\nu_{n+1} = \begin{pmatrix} & \nu_n \\ 1 & \end{pmatrix}, \quad n \in \mathbb{N}, \text{ where } \nu_1 = (1). \tag{1.3}$$

Denote by

$$\mathrm{O}_{2n}(\nu_{2n}) = \{ g \in \mathrm{GL}_{2n} \mid g^t \cdot \nu_{2n} \cdot g = \nu_{2n} \} \quad \text{and}$$
$$\mathrm{SO}_{2n}(\nu_{2n}) = \{ g \in \mathrm{O}_{2n} \mid \text{ and } \det g = 1 \}$$

the even orthogonal groups and even special orthogonal group with respect to the form ν_{2n}.

Define an operator $'$ on the set of all square matrices of any degree by

$$' : g \mapsto g' = \nu_n g^t \nu_n^{-1}, \quad \text{for } g \in \mathrm{Mat}_n.$$

Let $\mathrm{P} = \bar{\mathrm{M}}\bar{\mathrm{N}}$ be the **Siegel parabolic subgroup** of SO_{4n} consisting of elements of the following form:

$$(g, X) = \begin{pmatrix} g & 0 \\ 0 & \nu_{2n} g^{-t} \nu_{2n} \end{pmatrix} \begin{pmatrix} \mathrm{I}_{2n} & X \\ & \mathrm{I}_{2n} \end{pmatrix}, \quad g \in \mathrm{GL}_{2n} \text{ and } X' = -X. \tag{1.4}$$

The symplectic group $\mathrm{Sp}_{2n}(J_{2n})$ is given by

$$\mathrm{Sp}_{2n}(J_{2n}) = \{ g \in \mathrm{GL}_{2n} \mid g^t \cdot J_{2n} \cdot g = J_{2n} \}, \quad \text{where } J_{2n} = \begin{pmatrix} & \nu_n \\ -\nu_n & \end{pmatrix}.$$

The **generalized Shalika subgroup** $\mathcal{H}_{2n}$ of SO_{4n} was introduced in [JngQ07], which is the subgroup of P consisting of elements (g, X) with $g \in \mathrm{Sp}_{2n}(J_{2n})$. That is

$$\mathcal{H}_{2n} = \{ (g, X) \in \mathrm{P} \mid g \in \mathrm{Sp}_{2n}(J_{2n}) \}.$$

We write J for J_{2n}, $\mathrm{Sp}_{2n} = \mathrm{Sp}_{2n}(J)$ and $\mathcal{H} = \mathcal{H}_{2n}$, when n is understood. Define (**generalized Shalika character**) $\psi_{\mathcal{H}}$ of $\mathcal{H}_{2n}$ by letting

$$\psi_{\mathcal{H}}((g, X)) = \psi(\mathrm{tr}(J_{2n} X \nu_{2n})) = \psi(\mathrm{tr}(KX)), \quad \text{where } K = \begin{pmatrix} -\mathrm{I}_n & \\ & \mathrm{I}_n \end{pmatrix}. \tag{1.5}$$

Definition 1.2. An **anti-involution** is a degree two operator on algebras or groups, which reverses the order of factors.

Next, we introduce some anti-involutions which will play crucial roles in showing multiplicity freeness of generalized Shalika models.

Definition 1.3. Define an operator τ_n on GL_{4n} and SO_{4n} by

$$\tau_n : g \mapsto g^{\tau_n} = \epsilon g' \epsilon^{-1}, \quad \text{where } \epsilon = \mathrm{diag}(J_{2n}, -J_{2n}) \in \mathcal{H}_{2n}.$$

Define another operator J_{2n} on GL_{2n} and Sp_{2n} by

$$g^{J_{2n}} := -J_{2n} g^t J_{2n}.$$

We write $\tau = \tau_n$, when n is understood.

Lemma 1.4. *$'$, τ and J are all anti-involutions. Both $'$ and τ preserve $\mathcal{H}$. Furthermore, τ preserves $\psi_{\mathcal{H}}$ (i.e. $\psi(h^\tau) = \psi(h)$ for $h \in \mathcal{H}$).*

Lemma 1.5. *For $g \in \mathrm{Mat}_n$,*

$$\mathrm{tr}(gY) = 0, \quad \textit{for all } Y \textit{ satisfying } Y' = -Y \iff g' = g; \tag{1.6}$$

$$\mathrm{tr}(gY) = 0, \quad \textit{for all } Y \textit{ satisfying } Y^t = -Y \iff g^t = g. \tag{1.7}$$

For $g \in \mathrm{Mat}_{2n}$,

$$\mathrm{tr}(gY) = 0, \quad \textit{for all } Y \textit{ satisfying } Y^J = Y \iff g^J = -g. \tag{1.8}$$

Proof. We will only prove Eq. (1.8) and another two equivalent relations follow similar argument.

Note that

$$\mathrm{tr}\, T^J = \mathrm{tr}(-J T^t J) = \mathrm{tr}\, T, \quad \text{for all } T \in \mathrm{Mat}_{2n}.$$

Hence

$$\operatorname{tr}\big[\big(g+g^J\big)\big(X-X^J\big)\big]=\operatorname{tr}\big(gX-g^JX^J\big)-\operatorname{tr}\big(gX^J-g^JX\big)=0.$$

Then

$$\begin{aligned}&\operatorname{tr}\big(g+g^J\big)X=\operatorname{tr}\big(g+g^J\big)\frac{(X+X^T)+(X-X^t)}{2}=0,\quad \text{for all } X\in \mathrm{Mat}_{2n}\\ &\iff \operatorname{tr}\big[\big(g+g^J\big)Y\big]=0,\quad \text{for all } Y \text{ satisfying } Y^J=Y.\end{aligned}$$

Also, for $h\in \mathrm{Mat}_{2n}$ such that

$$\operatorname{tr}(hX)=0,\quad \text{for all } X\in \mathrm{Mat}_{2n} \iff h=0.$$

Therefore

$$\operatorname{tr}(gY)=0,\quad \text{for all } Y \text{ satisfying } Y^J=Y \iff g^J=-g. \qquad \square$$

The generalized Shalika character is well defined, since

Lemma 1.6. *An element $g\in \mathrm{GL}_{2n}$ satisfies*

$$\psi\big(\operatorname{tr}(KX)\big)=\psi\big(\operatorname{tr}(KgXg')\big),\quad \textit{for all } X \textit{ such that } X'=-X \tag{1.9}$$

if and only if $g\in \mathrm{Sp}_{2n}$.

Proof. Let $Y=XK$. Then $X=YK$ and $J=\nu_{2n}K$. Hence $X'=-X$ if and only if $Y^J=Y$. Note that $\operatorname{tr}(KX)=\operatorname{tr}(XK)=\operatorname{tr}Y$ and

$$\operatorname{tr}(KgXg')=\operatorname{tr}(gXg'K)=\operatorname{tr}(gYKg'K)=\operatorname{tr}\big(gYg^J\big)=\operatorname{tr}\big(g^JgY\big).$$

Hence Eq. (1.9) is equivalent to

$$\operatorname{tr}\big(g^JgY\big)=\operatorname{tr}(Y),\quad \text{for all } Y \text{ satisfying } Y^J=Y,$$

which is equivalent to

$$\operatorname{tr}\big(g^Jg-\mathrm{I}_{2n}\big)Y=0,\quad \text{for all } Y \text{ satisfying } Y^J=Y.$$

By Eq. (1.8), the above is equivalent to

$$g^Jg-\mathrm{I}_{2n}=-\big(g^Jg-\mathrm{I}_{2n}\big)^J \iff g^Jg=\mathrm{I}_{2n} \iff g\in \mathrm{Sp}_{2n}. \qquad \square$$

Definition 1.7. The *generalized Shalika functional* of an irreducible admissible representation V_ρ of SO_{4n} is a nonzero functional satisfying

$$\Lambda_\rho\big(\rho(h)v\big)=\psi_{\mathcal{H}_{2n}}(h)\Lambda_\rho(v),\quad \text{for all } s\in \mathcal{H}_{2n} \text{ and } v\in V_\rho. \tag{1.10}$$

We say that V_ρ has a **generalized Shalika model** if there exists a nontrivial generalized Shalika functional Λ_ρ satisfying the above equation.

Generalized Shalika models interact with other known models intensively. In [JngNQ], the authors proved that generalized Shalika models are disjoint with a series of degenerate Whittaker models. Also, it is shown in [JngQ07] that the Shalika model on $\mathrm{GL}_{2n}(\mathcal{F})$ and the generalized Shalika model on $\mathrm{SO}_{4n}(\mathcal{F})$ are compatible with respect to a unitarily parabolic induction.

For an irreducible, unitary, supercuspidal representation (τ, V_τ) of $\mathrm{GL}_{2n}(\mathcal{F})$, we consider the unitarily induced representation $\mathrm{I}(s,\tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ from the Siegel parabolic subgroup $\mathrm{P} = \bar{\mathrm{M}}\bar{\mathrm{N}}$, where the Levi part $\mathrm{M} \cong \mathrm{GL}_{2n}$. More precisely, a section $\phi_{\tau,s}$ in $\mathrm{I}(s,\tau)$ is a smooth function from $\mathrm{SO}_{4n}(\mathcal{F})$ to V_τ, such that

$$\phi_{\tau,s}\big(m(a)ng\big) = |\det a|^{\frac{s}{2}+\frac{2n-1}{2}}\tau(a)\phi_{\tau,s}(g),$$

where $m(a) \in \mathrm{M}$ with $a \in \mathrm{GL}_{2n}(\mathcal{F})$. In other words, one has

$$\mathrm{I}(s,\tau) = \mathrm{Ind}_{\mathrm{P}(\mathcal{F})}^{\mathrm{SO}_{4n}(\mathcal{F})}\big(|\det|^{\frac{s}{2}}\cdot\tau\big).$$

We recall from [JngQ07] the following result.

Theorem 1.8. *(See Theorem 3.1 [JngQ07].) The induced representation* $\mathrm{I}(s,\tau)$ *admits no nonzero generalized Shalika functionals except that* $s = 1$*. When* $s = 1$*,* $\mathrm{I}(1,\tau)$ *admits a nonzero generalized Shalika functional if and only if the supercuspidal datum* τ *admits a nonzero Shalika functional. In this case, the generalized Shalika functionals of* $\mathrm{I}(1,\tau)$ *are unique and the nonzero generalized Shalika functionals of* $\mathrm{I}(1,\tau)$ *must factor through the unique Langland's quotient* $\mathrm{J}(1,\tau)$*.*

The above theorem states the hereditary properties of induced representation between Shalika models of GL_{2n} and generalized Shalika models of SO_{4n} for irreducible, unitary, supercuspidal representations on GL_{2n}. We may wonder if Shalika models of GL_{2n} and generalized Shalika models of SO_{4n} share more common properties. One of the interesting questions in representation theory is multiplicity freeness. The multiplicity freeness of Shalika models of GL_{2n} is known, and the uniqueness of generalized Shalika models for SO_{4n} will be established in the following theorem.

Theorem 1.9. *Let* $\mathrm{SO}_{4n} = \mathrm{SO}(4n, F)$*, where* F *is either a finite field* $\mathbb{F}_q$ *with characteristic* p*,* $p \neq 2$*, or a* p*-adic field* $\mathcal{F}$*. Then*

$$\dim \mathrm{Hom}_{\mathrm{SO}_{4n}}\big(V_\pi, \mathrm{Ind}_{\mathcal{H}_{2n}}^{\mathrm{SO}(4n,F)}\psi_{\mathcal{H}_{2n}}\big) \leqslant 1, \quad n \in \mathbb{N}$$

for any irreducible admissible representation V_π *of* SO_{4n}*.*

2. Common strategy

Let G be a finite group and V_π be a representation of G. By Schur's lemma,

$$V_\pi \text{ is multiplicity free if and only if } \mathrm{Hom}_G(V_\pi, V_\pi) \cong (\mathbb{C})^k,$$

where k is the number of irreducible components of V_π. Then

$$V_\pi \text{ is multiplicity free if and only if the endomorphism algebra } \mathrm{Hom}_G(V_\pi, V_\pi) \text{ is abelian.}$$

Moreover, when $V_\pi = \mathrm{Ind}_H^G\rho$ is an induced representation, $\mathrm{Hom}_G(V_\pi, V_\pi)$ is explicitly characterized by Mackey theorem.

Theorem 2.1 *(Mackey's Theorem). Let G be a finite group, H_i its subgroups and π_i representations of H_i, $i = 1, 2$. Denote*

$$\mathfrak{S} = \left\{\Delta : G \mapsto \mathrm{Hom}_{\mathbb{C}}(\pi_1, \pi_2) \;\middle|\; \Delta(h_2 g h_1) = \pi_2(h_2) \circ \Delta(g) \circ \pi_1(h_1),\ h_i \in H_i\right\}.$$

As a vector space, $\mathrm{Hom}_G(\mathrm{Ind}_{H_1}^G \pi_1, \mathrm{Ind}_{H_2}^G \pi_2)$ is isomorphic to $\mathfrak{S}$. Given any $\Delta \in \mathfrak{S}$, the corresponding intertwining operator

$$T_\Delta \in \mathrm{Hom}_G\left(\mathrm{Ind}_{H_1}^G \pi_1, \mathrm{Ind}_{H_2}^G \pi_2\right)$$

is given by $T_\Delta(f_1) = \Delta * f_1$ for $f_1 \in \mathrm{Ind}_{H_1}^G \pi_1$, where the convolution

$$\Delta * f_1(x) = \frac{1}{|G|} \sum_{g \in G} \Delta\left(x g^{-1}\right) f_1(g).$$

Especially, when $H_1 = H_2, \pi_1 = \pi_2$, the algebra $\mathrm{Aut}_G(\mathrm{Ind}_{H_1}^G \pi_1)$ is isomorphic to $(\mathfrak{S}, \cdot)$, where the multiplication $\cdot$ is given by

$$\Delta_1 \cdot \Delta_2(g) = \sum_{x \in G} \Delta_1\left(g x^{-1}\right) \circ \Delta_2(x), \quad \Delta_i \in \mathfrak{S}.$$

In order to show that the endomorphism algebra is abelian, identifying an anti-involution to interchange the order of factors is a common strategy. If for some anti-involution σ on G,

$$T^\sigma = T, \quad \text{for all } T \in \mathrm{Aut}_G\left(\mathrm{Ind}_{H_1}^G \pi_1\right),$$

then

$$T_1 \circ T_2 = (T_1 \circ T_2)^\sigma = T_2^\sigma \circ T_1^\sigma = T_2 \circ T_1.$$

Hence $\mathrm{Aut}_G(\mathrm{Ind}_{H_1}^G \pi_1)$ is abelian. The analogue of this method in p-adic case is "Gelfand–Kazhdan criterion".

Let $C_c^\infty(X)$ denote the space of smooth, compactly supported functions on an l-adic space X (in the sense of [BZ76]). Let $\mathfrak{D}(X)$ denote the space of linear functionals on $C_c^\infty(X)$. Given a p-adic group G, define actions L_g and R_g on G; $C_c^\infty(G)$ and $\mathfrak{D}(G)$ as the following:

$$L_g \cdot x = gx; \qquad R_g \cdot x = x g^{-1};$$
$$(L_g \cdot f)(x) = f\left(g^{-1} x\right); \qquad (R_g \cdot f)(x) = f(xg);$$
$$(L_g \cdot T)(f) = T(L_{g^{-1}} \cdot f); \qquad (R_g \cdot T)(f) = T(R_{g^{-1}} \cdot f),$$

where $g, x \in G$; $f \in C_c^\infty(G)$ and $T \in \mathfrak{D}(G)$.

Theorem 2.2. *(See [GK75], or refer to [SZ08] for a more general version on Gelfand–Kazhdan criterion.) Let ψ and ψ^σ be characters of a closed unimodular subgroup H of G. Suppose that there is an anti-involution σ of G so that σ stabilizes H, $\psi(h^\sigma) = \psi^\sigma(h)$, and σ acts trivially on all distributions T so that*

$$T(L_h \eta) = \psi(h) \cdot T(\eta); \qquad T(R_h \eta) = \psi^\sigma(h)^{-1} \cdot T(\eta) \quad \textit{for } \eta \in C_c^\infty(G).$$

Then

$$\dim \operatorname{Hom}_G\big(\pi\,;\operatorname{Ind}_H^G\psi\big)\cdot\dim \operatorname{Hom}_H\big(\operatorname{Res}_H^G\tilde{\pi}\,;\psi^\sigma\big)\leqslant 1,$$

where π is any irreducible representation of G and $\tilde{\pi}$ its contragradient.

By the above argument, to prove multiplicity freeness amounts to showing the following lemma.[1]

Lemma 2.3 *(Key Lemma). For each $g\in \mathrm{SO}_{4n}$, there exist $h, r\in \mathcal{H}_{2n}$ (depending on g) such that one of the following conditions holds*:

Condition 1. $hgr^{-1}=g$, and $\psi_{\mathcal{H}_{2n}}(hr^{-1})\neq 1$.
Condition 2. $hgr^{-1}=g'$, and $\psi_{\mathcal{H}_{2n}}(hr^{-1})=1$.

Remark 2.4. If g satisfies Condition 1 (resp. 2), then g', g^{-1} and sgt, $s,t\in\mathcal{H}_{2n}$ will also satisfy the same condition (with different h,r). Hence we also say that a $\mathcal{H}\times\mathcal{H}$-double coset satisfies Condition 1 or 2 if "**one of its elements does**".

3. Proof of Lemma 2.3

Now we consider generalized Shalika model for SO_{4n} for an arbitrary fix $n\in\mathbb{N}$. Denote by $\tilde{g}$ the embedding of $g\in \mathrm{GL}_{2n}$ into SO_{4n}. More precisely,

$$\tilde{g}=\begin{pmatrix} g & \\ & g'^{-1}\end{pmatrix}\in \mathrm{SO}_{4n},\quad \text{for } g\in \mathrm{GL}_{2n}.$$

Through Bruhat decomposition, we can see a complete set of representatives for double cosets of $\mathcal{H}_{2n}\backslash \mathrm{SO}_{4n}/\mathcal{H}_{2n}$ can be chosen as $\widehat{\mathrm{GL}}_{2n}W\widehat{\mathrm{GL}}_{2n}$, where $\widehat{\mathrm{GL}}_{2n}$ is the diagonal embedding of $\mathrm{GL}_{2n}\hookrightarrow \mathrm{SO}_{4n}$ and W is the Weyl group of SO_{4n}. Similarly, $\widehat{\mathrm{Sp}}_{2n}$ will denote the diagonal embedding of Sp_{2n} into SO_{4n}. Moreover, a complete set of representatives for double cosets of $\mathcal{H}_{2n}\backslash \mathrm{SO}_{4n}/\mathcal{H}_{2n}$ can be chosen as

$$\{\tilde{p}w_k\tilde{g}\mid p,g\in \mathrm{GL}_{2n},\ 0\leqslant k\leqslant n\},$$

where

$$w_k=\left(\begin{array}{cc|cc} \mathrm{I}_{2k} & & & \\ & & \mathrm{I}_{2n-2k} & \\ \hline & \mathrm{I}_{2n-2k} & & \\ & & & \mathrm{I}_{2k}\end{array}\right),\quad 0\leqslant k\leqslant n.$$

Let U_{2n} denote the upper triangular maximal unipotent subgroup of GL_{2n} and $W(\mathrm{GL}_{2n})$ the Weyl group of GL_{2n}, identified with the group of permutation matrices in GL_{2n}. Here we introduce some results regarding admissibility and matrix identities, which will be used in the proof of Lemma 2.3.

Lemma 3.1. *(See [JR92b, Lemma 2].) Every nonsingular skew symmetric matrix of degree $2n$ can be written in the form*

$$s=u\sigma\lambda\, u^t$$

[1] The involution $\tau=\tau_n$ in Definition 1.3 will play the role of σ in Theorem 2.2. To shorten the computation and notation, we avoid work on τ directly and prove Lemma 2.3 with respect to involution $'$ first. Then Lemma 4.4 regarding similar conditions with $'$ replaced by τ follows.

with $u \in \mathrm{U}_{2n}$, λ is a diagonal matrix in GL_{2n}, $\sigma \in W(\mathrm{GL}_{2n})$ such that

$$\sigma^2 = 1, \quad \sigma\lambda\sigma^{-1} = -\lambda.$$

Proposition 3.2. *Let A be a nonsingular skew symmetric matrix of degree $2n$. Then there exists a $B \in \mathrm{GL}_{2n}$ such that*

$$BAB^t = -A. \tag{3.1}$$

Proof. Write $A = u\sigma\lambda\ u^t$ for some $u \in \mathrm{U}_{2n}$, and λ is a diagonal matrix in GL_{2n}, $\sigma \in W(\mathrm{GL}_{2n})$ such that

$$\sigma^2 = 1, \quad \sigma\lambda\sigma^{-1} = -\lambda.$$

Note that $\sigma = \sigma^t = \sigma^{-1}$, since $\sigma \in W(\mathrm{GL}_{2n})$ and $\sigma^2 = \mathrm{I}_{2n}$. Let $B = u\sigma u^{-1}$. Then

$$\begin{aligned} BAB^t &= \left(u\sigma u^{-1}\right)\left(u\sigma\lambda u^t\right)\left(u^{-t}\sigma^t u^t\right) = u\sigma^2\lambda\sigma^t u^t \\ &= u\left(\lambda\sigma^t\right)u^t = u(-\sigma\lambda)u^t = -A. \qquad \square \end{aligned}$$

Let D_{2n} be the set of diagonal matrix in GL_{2n}. Denote

$$\mathcal{A} = \left\{\eta = \sigma\lambda \;\middle|\; \lambda \in \mathrm{D}_{2n},\ \sigma \in W(\mathrm{GL}_{2n}),\ \sigma^2 = 1,\ \sigma\lambda\sigma^{-1} = -\lambda\right\}.$$

We recall the definition of monomial matrices.

Definition 3.3. If in every row and every column of $A \in \mathrm{GL}_n$ there is exactly one nonzero entry, then A is a **monomial matrix**. We denote the set of monomial matrix by Mo. Then $\mathcal{A} \subset \mathrm{Mo}$. We also call an $m \times n$ matrix **generalized monomial matrix** if there is at most one nonzero entry in each row and each column. Denote by GMo the set of generalized monomial matrices.

To simplify the calculation of admissibility of double cosets, we extend the notion of generalized Shalika groups as follows. Let A be any nonsingular skew symmetric matrix of degree $2n$. Define

$$\mathcal{H}_A = \widehat{\mathrm{Sp}}_{2n}(A)\bar{N},$$

where elements in $\mathcal{H}_A$ are $(h, X) \in P$ with $h \in \mathrm{Sp}_{2n}(A)$, and

$$\mathrm{Sp}_{2n}(A) = \left\{x \in \mathrm{GL}_{2n} \;\middle|\; x^t A x = A\right\}.$$

Let

$$g \in \mathrm{GL}_{2n} \quad \text{satisfying } A = g^{-t} J_{2n} g^{-1}.$$

Then

$$\mathrm{Sp}_{2n}(A) = g\mathrm{Sp}_{2n}(J_{2n})g^{-1} \quad \text{and} \quad g\mathcal{H}g^{-1} = \mathcal{H}_A.$$

Define a character $\psi_{\mathcal{H}_A}$ on $\mathcal{H}_A$ by

$$\psi_{\mathcal{H}_A}(h, X) = \psi\left(\mathrm{tr}(AX\nu_{2n})\right), \quad (h, X) \in \mathcal{H}_A. \tag{3.2}$$

It is well defined, since

$$\begin{aligned}\psi_{\mathcal{H}_A}\big((h,0)(1,X)\big(h^{-1},0\big)\big) &= \psi_{\mathcal{H}_A}(1, hX\,h') \\ &= \psi\big(\mathrm{tr}\big(AhX\nu_{2n}\,h^t\big)\big) \\ &= \psi\big(\mathrm{tr}\big(h^t AhX\nu_{2n}\big)\big) \\ &= \psi\big(\mathrm{tr}(A\,X\nu_{2n})\big) \quad \text{for } h \in \mathrm{Sp}_{2n}(A).\end{aligned}$$

When $A = J_{2n}$, $\mathcal{H}_A = \mathcal{H}_{2n}$ is the generalized Shalika group as defined before, and $\psi_{\mathcal{H}_A} = \psi_{\mathcal{H}_{2n}}$.

Definition 3.4. For A, B both skew symmetric, we say that a double coset $\mathcal{H}_B w \mathcal{H}_A$, $w \in \mathrm{SO}_{4n}(F)$ is **admissible** if

$$\psi_{\mathcal{H}_A}\big((h,X)\big) = \psi_{\mathcal{H}_B}\big(w(h,X)w^{-1}\big), \quad \text{for all } (h,X) \in w^{-1}\mathcal{H}_B w \cap \mathcal{H}_A. \tag{3.3}$$

Remark 3.5. If $g \in \mathrm{SO}_{4n}$ satisfies Condition 1, then

$$hgr^{-1} = g \quad \text{and} \quad \psi_{\mathcal{H}}\big(hr^{-1}\big) \neq 1, \quad \text{for some } h, r \in \mathcal{H}.$$

Then

$$g^{-1}hg = r \in \mathcal{H} \cap g^{-1}\mathcal{H}g \quad \text{and} \quad \psi_{\mathcal{H}}(h) \neq \psi_{\mathcal{H}}(r).$$

Therefore $g \in \mathrm{SO}_{4n}$ satisfies Condition 1 if and only if $\mathcal{H}g\mathcal{H}$ is not admissible.

Lemma 3.6. *The double coset $\mathcal{H}_B w \mathcal{H}_A$ is admissible if and only if $\mathcal{H}\tilde{p}w\tilde{g}\mathcal{H}$ is admissible, where $g, p \in \mathrm{GL}_{2n}$ and $A = g^{-t}J_{2n}g^{-1}$, $B = p^t J_{2n} p$.*

Lemma 3.7. *For $1 \leqslant k \leqslant n-1$, to consider the admissibility of all $\mathcal{H}\tilde{p}w_k\tilde{g}\mathcal{H}$, $p, g \in \mathrm{GL}_{2n}$, it suffices to consider the admissibility of all double cosets in the form of*

$$\mathcal{H}_{\eta_2} w_k \tilde{u}_1 \mathcal{H}_{\eta_1}, \quad \text{with } u_1 = \begin{pmatrix} v_1 & \\ & v_3 \end{pmatrix}, \; v_1 \in \mathrm{U}_{2k}, \; v_3 \in \mathrm{U}_{2(n-k)}, \; \eta_1, \eta_2 \in \mathcal{A}. \tag{3.4}$$

Or, equivalently we may consider the admissibility of all double cosets in the form of

$$\mathcal{H}_{\eta_2} \tilde{u}_2 w_k \tilde{u}_1 \mathcal{H}_{\eta_1}, \quad \text{with } u_1 = \begin{pmatrix} \mathrm{I}_{2k} & \\ & v_3 \end{pmatrix}, \; u_2 = \begin{pmatrix} v_4 & \\ & \mathrm{I}_{2n-2k} \end{pmatrix}, \tag{3.5}$$

where $v_4 \in \mathrm{U}_{2k}$, $v_3 \in \mathrm{U}_{2(n-k)}$, $\eta_1, \eta_2 \in \mathcal{A}$.

Proof. For $1 \leqslant k \leqslant n-1$, consider any representative

$$\mathcal{H}\tilde{p}w_k\tilde{g}\mathcal{H}, \quad \text{where } \tilde{p}, \tilde{g} \in \widehat{\mathrm{GL}}_{2n}.$$

Let $B = p^t J_{2n} p$, $A = g^{-t} J_{2n} g^{-1}$. By Lemma 3.1, there exist $u_1, u_2 \in \mathrm{U}_{2n}$, $\eta_1, \eta_2 \in \mathcal{A}$ such that $B = u_2^t \eta_2 u_2$, and $A = u_1^{-t}\eta_1 u_1^{-1}$. Since $\mathcal{H}\tilde{p}w_k\tilde{g}\mathcal{H}$ is admissible if and only if $\mathcal{H}_{\eta_2}\tilde{u}_2 w_k \tilde{u}_1 \mathcal{H}_{\eta_1}$ is admissible, we may work on $\mathcal{H}_{\eta_2}\tilde{u}_2 w_k \tilde{u}_1 \mathcal{H}_{\eta_1}$ instead. For fixed k, $1 \leqslant k \leqslant n-1$, write

$$u_1 = \begin{pmatrix} v_1 & v_2 \\ & v_3 \end{pmatrix}; \qquad u_2 = \begin{pmatrix} v_4 & v_5 \\ & v_6 \end{pmatrix},$$

where $v_1, v_4 \in \mathrm{U}_{2k},\ v_2, v_5 \in \mathrm{Mat}_{2k\times 2(n-k)},\ v_3, v_6 \in \mathrm{U}_{2(n-k)}$. Then

$$w_k^{-1}\tilde{u}_2 w_k = w_k^{-1}\left(\begin{array}{cc|cc} v_4 & v_5 & & \\ & v_6 & & \\ \hline & & v_6^* & v_5^* \\ & & & v_4^* \end{array}\right) w_k = \left(\begin{array}{cc|cc} v_4 & & v_5 & \\ & v_6^* & & v_5^* \\ \hline & & v_6 & \\ & & & v_4^* \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} v_4 & & & \\ & v_6^* & & \\ \hline & & v_6 & \\ & & & v_4^* \end{array}\right)\left(\begin{array}{cc|cc} \mathrm{I}_{2k} & & v_4^{-1}v_5 & \\ & \mathrm{I}_{2(n-k)} & & v_6^{-*}v_5^* \\ \hline & & \mathrm{I}_{2(n-k)} & \\ & & & \mathrm{I}_{2k} \end{array}\right).$$

Since

$$\left(\begin{array}{cc|cc} \mathrm{I}_{2k} & & v_4^{-1}v_5 & \\ & \mathrm{I}_{2(n-k)} & & v_6^{-*}v_5^* \\ \hline & & \mathrm{I}_{2(n-k)} & \\ & & & \mathrm{I}_{2k} \end{array}\right) \in \bar{N}$$

(symmetrically, we can take representative u_1 with v_2 part trivial) and $\begin{pmatrix} v_4 & \\ & v_6^* \end{pmatrix} \in \mathrm{U}_{2n}$, by replacing u_1 with $u_1' = \begin{pmatrix} v_4 & \\ & v_6^* \end{pmatrix} u_1$,

$$\mathcal{H}_{\eta_2}\tilde{u}_2 w_k \tilde{u}_1 \mathcal{H}_{\eta_1} = \mathcal{H}_{\eta_2} w_k \tilde{u}_1' \mathcal{H}_{\eta_1}.$$

Now it suffices to consider the admissibility of double cosets in the form

$$\mathcal{H}_{\eta_2} w_k \tilde{u}_1 \mathcal{H}_{\eta_1}, \quad \text{with } u_1 = \begin{pmatrix} v_1 & \\ & v_3 \end{pmatrix},\ v_1 \in \mathrm{U}_{2k},\ v_3 \in \mathrm{U}_{2(n-k)},\ \eta_1, \eta_2 \in \mathcal{A}.$$

Or, equivalently we may consider the admissibility of

$$\mathcal{H}_{\eta_2}\tilde{u}_2 w_k \tilde{u}_1 \mathcal{H}_{\eta_1}, \quad \text{with } u_1 = \begin{pmatrix} \mathrm{I}_{2k} & \\ & v_3 \end{pmatrix},\ u_2 = \begin{pmatrix} v_4 & \\ & \mathrm{I}_{2n-2k} \end{pmatrix},$$

where $v_4 \in \mathrm{U}_{2k},\ v_3 \in \mathrm{U}_{2(n-k)},\ \eta_1, \eta_2 \in \mathcal{A}$. $\square$

Before we prove Lemma 2.3, we cite two auxiliary propositions.

Proposition 3.8. *(See the proof of Corollary 3.2 [GoGu07].) For any $g \in \mathrm{GL}_{2n}(F)$, there exist $Q_1, Q_2 \in \mathrm{Sp}_{2n}(J')$, such that*

$$g^t = Q_1 g Q_2,$$

where the symplectic form J' is given by $\begin{pmatrix} & \mathrm{I}_n \\ -\mathrm{I}_n & \end{pmatrix}$.

Proposition 3.9. *For any $g \in \mathrm{GL}_{2n}$, there exist $P_1, P_2 \in \mathrm{Sp}_{2n}(J)$ satisfying*

$$g' = \nu_{2n} g^t \nu_{2n} = P_1 g P_2.$$

Proof. For any $g \in \mathrm{GL}_{2n}$, let $Q_1, Q_2 \in \mathrm{Sp}_{2n}(J')$ be such that

$$\begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix} g^t \begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix} = Q_1 \left(\begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix} g \begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix} \right) Q_2.$$

Then

$$\begin{aligned} g' &= \begin{pmatrix} \mathrm{I}_n & \\ & \nu_n \end{pmatrix} \left(\begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix} g^t \begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix} \right) \begin{pmatrix} \mathrm{I}_n & \\ & \nu_n \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{I}_n & \\ & \nu_n \end{pmatrix} \left(Q_1 \begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix} g \begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix} Q_2 \right) \begin{pmatrix} \mathrm{I}_n & \\ & \nu_n \end{pmatrix} \\ &= P_1 g P_2, \end{aligned}$$

where $P_1 = \begin{pmatrix} \mathrm{I}_n & \\ & \nu_n \end{pmatrix} Q_1 \begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix}$, $P_2 = \begin{pmatrix} \nu_n & \\ & \mathrm{I}_n \end{pmatrix} Q_2 \begin{pmatrix} \mathrm{I}_n & \\ & \nu_n \end{pmatrix} \in \mathrm{Sp}_{2n}(J)$. $\square$

To show that Lemma 2.3 holds, it suffices to show that any admissible double coset satisfies Condition 2, and we will replace Condition 2 with a stronger one, Condition 3.

Condition 3. We say that a double coset $\mathcal{H}\lambda\mathcal{H}$ satisfies Condition 3, if there exists a representative γ such that $s_1 \gamma s_2 = \gamma'$, for some $s_1, s_2 \in \widehat{\mathrm{Sp}}_{2n}$.

Then the following lemma implies Lemma 2.3.

Lemma 3.10. *Any admissible double coset satisfies Condition* 3.

Proof. We proceed by induction on $n \in \mathbb{N}$ of SO_{4n} to show that any admissible double coset satisfies Condition 3. For $n = 1$, the case of SO(4), $\mathrm{Sp}_2 = \mathrm{SL}_2$, there are two kinds of representatives for the double cosets of $\mathcal{H}_2 \backslash \mathrm{SO}(4)/\mathcal{H}_2$: $\widehat{\mathrm{GL}}_{2n} w_0 \widehat{\mathrm{GL}}_{2n}$ and $\widehat{\mathrm{GL}}_{2n} w_1 \widehat{\mathrm{GL}}_{2n}$.

(1) For $k = 0$, $w_0 = \begin{pmatrix} & \mathrm{I}_2 \\ \mathrm{I}_2 & \end{pmatrix}$. Representatives can be chosen to be in the form of

$$\begin{pmatrix} & g \\ (g')^{-1} & \end{pmatrix}, \quad \text{with } g \in \mathrm{GL}_2.$$

Then by Proposition 3.9, there exist $P_1, P_2 \in \mathrm{Sp}_2$ such that

$$g' = P_1 g P_2.$$

Hence

$$\begin{pmatrix} & g \\ (g')^{-1} & \end{pmatrix}' = \begin{pmatrix} P_1 & \\ & (P_1')^{-1} \end{pmatrix} \begin{pmatrix} & g \\ (g')^{-1} & \end{pmatrix} \begin{pmatrix} (P_2')^{-1} & \\ & P_2 \end{pmatrix},$$

and $\begin{pmatrix} & g \\ (g')^{-1} & \end{pmatrix}$ satisfies Condition 3.

(2) For $k = 1$, $w_1 = \mathrm{I}_4$, representatives can be chosen to be in the form $\tilde{g}$, with $g \in \mathrm{GL}_2/\mathrm{Sp}_2 = \{\mathrm{diag}(a, 1) \mid a \in \mathcal{F}^*\}$. Let

$$n_1(x) = \begin{pmatrix} 1 & & x & \\ & 1 & & -x \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Then

$$n_2(x)=\tilde{g}n_1\tilde{g}^{-1}=\begin{pmatrix}1&&ax&\\&1&&-ax\\&&1&\\&&&1\end{pmatrix},$$

and $\psi_{\mathcal{H}}(n_1)=\psi(-2x)$, $\psi_{\mathcal{H}}(n_2)=\psi(-2ax)$. Hence $\mathcal{H}\tilde{g}\mathcal{H}$ is admissible only when $a=1$. As for $a=1$, $\tilde{g}=\mathrm{I}_4$, it apparently satisfies Condition 3.

Now we assume Condition 3 holds for every admissible double coset $\mathcal{H}\lambda\mathcal{H}$ in $\mathrm{SO}(4n-4)$.

In the case of SO_{4n}, there are $n+1$ kinds of representatives for $\mathcal{H}_{2n}\backslash\mathrm{SO}_{4n}/\mathcal{H}_{2n}$, and they are $\widehat{\mathrm{GL}}_{2n}w_k\widehat{\mathrm{GL}}_{2n}$, $0\leqslant k\leqslant n$. We discuss them in three setups.

(1) For $k=0$, $w_0=\left(\begin{smallmatrix}&\mathrm{I}_{2n}\\\mathrm{I}_{2n}&\end{smallmatrix}\right)$. Representatives can be chosen to be in the form of

$$\begin{pmatrix}&g\\(g')^{-1}&\end{pmatrix},\quad\text{with } g\in\mathrm{GL}_{2n}.$$

By the same argument as in the first case of SO(4), we can see that $\left(\begin{smallmatrix}&g\\g^{-\prime}&\end{smallmatrix}\right)$ satisfies Condition 3.

(2) For $k=n$, $w_n=\mathrm{I}_{4n}$, representatives can be chosen to be in the form $\tilde{g}$, with $g\in\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$. Let

$$n_1(X)=\begin{pmatrix}\mathrm{I}_{2n}&X\\&\mathrm{I}_{2n}\end{pmatrix},\quad\text{where } X'=-X.$$

Then

$$n_2(X)=\tilde{g}n_1\tilde{g}^{-1}=\begin{pmatrix}\mathrm{I}_{2n}&gXg'\\&\mathrm{I}_{2n}\end{pmatrix}$$

and $\psi_{\mathcal{H}}(n_1(X))=\psi(\mathrm{tr}(KX))$, $\psi_{\mathcal{H}}(n_2(X))=\psi(\mathrm{tr}(KgXg'))$. By Lemma 1.6, $\mathcal{H}\tilde{g}\mathcal{H}$ is admissible if and only if $g\in\mathrm{Sp}_{2n}$. In the case of $g\in\mathrm{Sp}_{2n}$, $\tilde{g}$ apparently satisfies Condition 3.

(3) For $1\leqslant k\leqslant n-1$, consider any representative in the form of $\mathcal{H}\tilde{p}w_k\tilde{g}\mathcal{H}$, $p,g\in\mathrm{GL}_{2n}$. By Lemma 3.7, we may work on

$$\mathcal{H}_{\eta_2}\tilde{u}_2w_k\tilde{u}_1\mathcal{H}_{\eta_1},\quad\text{for some } u_1=\begin{pmatrix}\mathrm{I}_{2k}&\\&v_3\end{pmatrix},\ u_2=\begin{pmatrix}v_4&\\&\mathrm{I}_{2k}\end{pmatrix},\ \eta_1,\eta_2\in\mathcal{A},$$

such that $p^tJp=u_2^t\eta_2u_2:=\theta_2$, $g^{-t}Jg^{-1}=u_1^{-t}\eta_1u_1^{-1}:=\theta_1$.

We assume that $\mathcal{H}\tilde{p}w_k\tilde{g}\mathcal{H}$ is admissible, which is equivalent to $\mathcal{H}_{\eta_2}\tilde{u}_2w_k\tilde{u}_1\mathcal{H}_{\eta_1}$ or $\mathcal{H}_{\theta_2}w_k\mathcal{H}_{\theta_1}$ is admissible. Write $j=2n-2k$. Let

$$n_1=n_1(Z)=\begin{pmatrix}\mathrm{I}_{2k}&&&Z\\&\mathrm{I}_j&&\\&&\mathrm{I}_j&\\&&&\mathrm{I}_{2k}\end{pmatrix},\quad\text{where } Z\in\mathrm{Mat}_{2k},\ Z'=-Z.$$

Then $n_1\in\bar{N}$ and

$$w_k\,n_1(w_k)^{-1}=n_1.$$

By admissibility,

$$\psi_{\mathcal{H}_{\theta_2}}\big(n_1(Z)\big) = \psi_{\mathcal{H}_{\theta_1}}\big(n_1(Z)\big), \quad \text{for all } Z \text{ satisfying } Z' = -Z.$$

That is

$$\psi\left(\operatorname{tr}\left(\theta_2 \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \nu_{2n}\right)\right) = \psi\left(\operatorname{tr}\left(\theta_1 \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \nu_{2n}\right)\right),$$

for all Z satisfying $Z' = -Z$. Write

$$\theta_2 = \begin{pmatrix} \alpha_{2k\times 2k} & \beta_{2k\times j} \\ \gamma_{j\times 2k} & \delta_{j\times j} \end{pmatrix}, \qquad \theta_1 = \begin{pmatrix} a_{2k\times 2k} & b_{2k\times j} \\ c_{j\times 2k} & d_{j\times j} \end{pmatrix},$$

where subscripts denote the sizes of the matrices. Since

$$\operatorname{tr}\left(\theta_2 \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \nu_{2n}\right) = \operatorname{tr}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \nu_{2n}\right) = \operatorname{tr}(\alpha Z \nu_{2k}),$$

now we have

$$\psi\big(\operatorname{tr}(\alpha Z \nu_{2k})\big) = \psi\big(\operatorname{tr}(a Z \nu_{2k})\big), \quad \text{for all } Z \text{ satisfying } Z' = -Z. \tag{3.6}$$

Let $Z\nu_{2k} = Y$, then $Y^t = -Y$. Eq. (3.6) is equivalent to

$$\psi\big(\operatorname{tr}(\alpha - a)Y\big) = 0, \quad \text{for all } Y \text{ satisfying } Y^t = -Y. \tag{3.7}$$

Since both α and a are skew symmetric, we obtain $\alpha = a$ by Lemma 1.5.

Let

$$n_3 = n_3(X) = \begin{pmatrix} \mathrm{I}_{2k} & & X & \\ & \mathrm{I}_j & & -X' \\ & & \mathrm{I}_j & \\ & & & \mathrm{I}_{2k} \end{pmatrix}, \quad \text{where } X \in \mathrm{Mat}_{2k\times j}.$$

Let

$$n_4 = n_4(X) = w_k n_3 (w_k)^{-1} = \begin{pmatrix} \mathrm{I}_{2k} & X & & \\ & \mathrm{I}_j & & \\ & & \mathrm{I}_j & -X' \\ & & & \mathrm{I}_{2k} \end{pmatrix}.$$

Admissibility guarantees that

$$\text{if } \begin{pmatrix} \mathrm{I}_{2k} & X \\ & \mathrm{I}_j \end{pmatrix} \in \mathrm{Sp}(\theta_2), \quad \text{then } \psi\left(\operatorname{tr}\left(\theta_1 \begin{pmatrix} X & 0 \\ 0 & -X' \end{pmatrix} \nu_{2n}\right)\right) = 1. \tag{3.8}$$

Note that $\left(\begin{smallmatrix} \mathrm{I}_{2k} & X \\ & \mathrm{I}_j \end{smallmatrix}\right) \in \mathrm{Sp}(\theta_2)$ if and only if

$$\alpha X = 0 \quad \text{and} \quad X^t\beta = -\gamma X = \big(X^t\beta\big)^t. \tag{3.9}$$

Write $X^t\beta = Y$, then $Y^t = Y$ and

$$\operatorname{tr}\left(\theta_1\begin{pmatrix} X & 0 \\ 0 & -X' \end{pmatrix}\nu_{2n}\right) = \operatorname{tr}\left(\nu_{2n}\begin{pmatrix} a_{2k\times 2k} & b_{2k\times j} \\ c_{j\times 2k} & d_{j\times j} \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & -X' \end{pmatrix}\right)$$
$$= \operatorname{tr}(\nu_j cX - \nu_{2k} bX') = 2\operatorname{tr}(-\nu_{2k} bX'). \tag{3.10}$$

Consider the following two cases:

(a) If $a = \alpha = 0$, then $2k = j = n$ and b, c, β, γ are of full rank. Then we have

$$\psi\big(2\operatorname{tr}\big(-\nu_{2k} b\nu_{2k} Y\beta^{-1}\nu_{2k}\big)\big) = \psi\big(2\operatorname{tr}\big(-\beta^{-1} b\nu_{2k} Y\big)\big) = 1, \tag{3.11}$$

for all Y satisfying $Y^t = Y$, by Lemma 1.5 which is equivalent to

$$\big(\beta^{-1} b\nu_{2k}\big)^t = -\beta^{-1} b\nu_{2k}. \tag{3.12}$$

Next we go back to work on the double coset $\mathcal{H}\tilde{p}w_k\tilde{g}\mathcal{H}$, where $p^t Jp = \theta_2$, and $\tilde{g}^{-t} J\tilde{g}^{-1} = \theta_1$. Then $g = \left(\begin{smallmatrix} g_1 & \\ & g_2 \end{smallmatrix}\right)$, and $p = \left(\begin{smallmatrix} p_1 & \\ & p_2 \end{smallmatrix}\right)$, for some $g_i, p_i, \in \mathrm{GL}_{2k}$. Representative of the double coset $\mathcal{H}\tilde{p}w_k\tilde{g}\mathcal{H}$

$$\tilde{p}w_k\tilde{g} = \left(\begin{array}{cc|cc} p_1 g_1 & & & \\ & & p_2(g_2')^{-1} & \\ \hline & (p_2')^{-1} g_2 & & \\ & & & (p_1')^{-1}(g_1')^{-1} \end{array}\right)$$

can be replace by $w_k\tilde{h}$, where

$$h = \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}, \quad \text{with } h_1 = p_1 g_1,\ h_2 = \big(p_2'\big)^{-1} g_2.$$

Set

$$\pi_k = w_k\tilde{h} = \left(\begin{array}{cc|cc} h_1 & & & \\ & & (h_2')^{-1} & \\ \hline & h_2 & & \\ & & & (h_1')^{-1} \end{array}\right).$$

Then

$$\pi_k' = \left(\begin{array}{cc|cc} h_1^{-1} & & & \\ & & h_2^{-1} & \\ \hline & h_2' & & \\ & & & h_1' \end{array}\right).$$

Now we claim that there exist

$$T_1 = \begin{pmatrix} t_1 & \\ & t_1^{-\prime} \end{pmatrix}, \qquad T_2 = \begin{pmatrix} t_2 & \\ & t_2^{-\prime} \end{pmatrix} \in \mathrm{Sp}_{4k}, \quad \text{such that} \quad \tilde{T}_1\pi_k\tilde{T}_2 = \pi_k'. \tag{3.13}$$

Eq. (3.13) is equivalent to find $t_1, t_2 \in \mathrm{GL}_{2k}$ such that

$$\begin{cases} t_1 h_1 t_2 = h_1^{-1} \\ (t_1')^{-1}(h_2')^{-1} t_2 = h_2^{-1} \end{cases} \iff \begin{cases} t_2 = h_1^{-1} t_1^{-1} h_1^{-1} \\ t_2^{-1} h_2' t_1' = h_2 \end{cases} \tag{3.14}$$

$$\iff \begin{cases} t_2 = h_1^{-1} t_1^{-1} h_1^{-1} \\ h_1 t_1 h_1 h_2' t_1' = h_2. \end{cases} \tag{3.15}$$

Since

$$\theta_2 = J, \qquad \theta_1 = h^{-t} J h^{-1} = \begin{pmatrix} & h_1^{-t} \nu_{2k} h_2^{-1} \\ -(h_1^{-t} \nu_{2k} h_2^{-1})^t & \end{pmatrix},$$

hence

$$b = h_1^{-t} \nu_{2k} h_2^{-1} \quad \text{and} \quad \beta = \nu_{2k}.$$

Eq. (3.12) becomes

$$b^t = -b. \tag{3.16}$$

By Proposition 3.2, there exists $H \in \mathrm{GL}_{2k}$ such that

$$H b^{-t} H^t = -b^{-t} = b^{-1}.$$

Let

$$t_1 = h_1^{-1} H \quad \text{and} \quad t_2 = h_1^{-1} t_1^{-1} h_1^{-1}.$$

Then $b^{-t} = h_1 \nu_{2k} h_2^t$ and

$$\begin{aligned} h_1 t_1 h_1 h_2' t_1' &= H\bigl(h_1 \nu_{2k} h_2^t\bigr) t_1^t \nu_{2k} = \bigl(H b^{-t} H^t\bigr)\bigl((h_1)^{-t} \nu_{2k}\bigr) \\ &= b^{-1}(b h_2) = h_2. \end{aligned}$$

Then Eq. (3.15) holds and Condition 3 holds for π_k.

(b) In the case $a = \alpha \neq 0$, let $a_{i_0, j_0} = \alpha_{i_0, j_0} \neq 0$ for some $1 \leqslant i_0, j_0 \leqslant 2k$. For any matrix g, we write $g = (g_{i,j})$ for its entries. Recall from (3.5)

$$\theta_2 = u_2^t \eta_2 u_2, \qquad \theta_1 = u_1^{-t} \eta_1 u_1^{-1},$$

where $u_1 = \begin{pmatrix} \mathrm{I}_{2k} & \\ & v_3 \end{pmatrix}$, $u_2 = \begin{pmatrix} v_4 & \\ & \mathrm{I}_{2k} \end{pmatrix}$, $\eta_1, \eta_2 \in \mathcal{A}$. Write

$$\eta_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad \eta_2 = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$

Then

$$\begin{aligned} \theta_1 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A & B v_3^{-1} \\ v_3^{-t} C & v_3^{-t} D v_3^{-1} \end{pmatrix}, \\ \theta_2 &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} v_4^t P v_4 & v_4^t Q \\ R v_4 & S \end{pmatrix}. \end{aligned} \tag{3.17}$$

Since

$$a = \alpha = A = v_4^t P v_4 \quad \text{with } a_{i_0,j_0} \neq 0,$$

then $A = P$ by Lemma 3.11 (see the next lemma) and the i_0-row of B (respectively Q) is zero. Then $b_{i_0,\kappa} = 0$ for all κ, since $b = Bv_3^{-1}$. Note that

$$\mathcal{H}_{\eta_2} u_2 w_k u_1 \mathcal{H}_{\theta_1} = \mathcal{H}_{\eta_2} u_2 w_k u_1 n_3(X) \mathcal{H}_{\theta_1},$$

where

$$n_3(X) = \begin{pmatrix} \mathrm{I}_{2k} & & X & \\ & \mathrm{I}_j & & -X' \\ & & \mathrm{I}_j & \\ & & & \mathrm{I}_{2k} \end{pmatrix} \in \bar{N}, \quad X \in \mathrm{Mat}_{2k \times j}.$$

Since

$$\tilde{u}_2 w_k \tilde{u}_1 n_3(X) = \tilde{u}_2 \tilde{n}_4(X) w_k \tilde{u}_1, n_4(X) = \begin{pmatrix} \mathrm{I}_{2k} & X v_3^{-1} \\ & \mathrm{I}_j \end{pmatrix},$$
$$\mathcal{H}_{\theta_3} w_k \mathcal{H}_{\theta_1} = \mathcal{H}_{\theta_2} w_k \mathcal{H}_{\theta_1},$$

where

$$\theta_3 = \big(n_4(X)\big)^t \theta_2 n_4(X) = \begin{pmatrix} v_4^t P v_4 & v_4^t Q + (v_4^t P v_4) X v_3^{-1} \\ * & * \end{pmatrix}.$$

The i_0-th row of $v_4^t P v_4$ is nonzero, so we may assume that the i_0-th row of $v_4^t Q + (v_4^t P v_4) X v_3^{-1}$ is zero by choosing suitable X. Similar argument holds for the j_0-row and we may assume that the i_0, j_0-th rows and columns of θ_3 and θ_1 have their only one nonzero term (respectively) appearing in $a = \alpha$.

By deleting the i_0, j_0, $4n+1$-i_0, $(4n+1$-$j_0)$-th rows and columns of w_k, we obtain a $\check{w}_k \in \mathrm{SO}(4n-4)$. Let $\check{\theta}_1$ (respectively $\check{\theta}_3$) be the skew symmetric matrix of size $n-2$ obtained by deleting the i_0, j_0-th rows and columns from θ_1 (respectively θ_3). Then any element $f \in \mathrm{Sp}_{\check{\theta}_1}$ (respectively $f \in \mathrm{Sp}_{\check{\theta}_3}$) can be embedded as $\underline{f} \in \mathrm{Sp}_{\theta_1}$ (respectively Sp_{θ_3}) by adding the i_0, j_0-th rows and columns such that the entries in the i_0, j_0-th rows and columns of $\underline{f}$ are all zeros except

$$\underline{f}_{i_0,i_0} = 1 \quad \text{and} \quad \underline{f}_{j_0,j_0} = 1.$$

This embedding gives a nature embedding of $\mathcal{H}_{\check{\theta}_3} \check{w}_k \mathcal{H}_{\check{\theta}_1}$ into $\mathcal{H}_{\theta_3} w_k \mathcal{H}_{\theta_1}$ and it respects the involution $'$ on SO_{4n} and SO_{4n-4}. Also, the admissibility of $\mathcal{H}_{\check{\theta}_3} \check{w}_k \mathcal{H}_{\check{\theta}_1}$ follows the same property of $\mathcal{H}_{\theta_3} w_k \mathcal{H}_{\theta_1}$. Note,

$$\text{if } p, q \text{ satisfy } p^t J p = q^t J q, \text{ then } pq^{-1} \in \mathrm{Sp}_J \text{ and } \mathcal{H}\tilde{p} = \mathcal{H}\tilde{q}.$$

Hence

$$\mathcal{H}\tilde{p}_1 w_k \tilde{g}_1 \mathcal{H} = \mathcal{H}\tilde{p}_2 w_k \tilde{g}_2 \mathcal{H}$$

for all $p_i^t J p_i = \theta_3$ and $g_i^{-t} J g_i^{-1} = \theta_1$, $i = 1, 2$.

Set $p_0, g_0 \in \mathrm{SO}_{4n-4}$ such that

$$(p_0)^t J(p_0) = \check{\theta}_3 \quad \text{and} \quad (g_0)^{-t} J(g_0)^{-1} = \check{\theta}_1.$$

Then $\mathcal{H}\tilde{p}_0\check{w}_k\tilde{g}_0\mathcal{H} \subset \mathrm{SO}_{4n-4}$ is admissible, since $\mathcal{H}_{\check{\theta}_3}\check{w}_k\mathcal{H}_{\check{\theta}_1}$ is admissible. By induction assumption, $\mathcal{H}\tilde{p}_0\check{w}_k\tilde{g}_0\mathcal{H}$ satisfies Condition 3, and then so does its embedding in $\mathcal{H}\tilde{p}w_k\tilde{g}\mathcal{H}$. Therefore, $\mathcal{H}\tilde{p}w_k\tilde{g}\mathcal{H}$ satisfies Condition 3. □

Lemma 3.11. *Assume that $A, P \in \mathrm{Mat}_k$ are skew symmetric and generalized monomial matrices such that $u^t P u = A$ for some $u \in \mathrm{U}_k$. Then $A = P$.*

Proof. We will proceed by induction. For $k = 2$, either $P = 0$ or $P = \left(\begin{smallmatrix} 0 & s \\ -s & 0 \end{smallmatrix}\right)$. In either case, it is easy to see that $A = P$ if $u^t P u = A$ for some $u \in \mathrm{U}_2$.

Next we assume that the conclusion is true for matrices of size n such that $2 \leqslant n < k$. Let $P = (P_{i,j})$, $A = (A_{i,j})$.

(1) If the first row of P is zero, then so is its first column by skew symmetry. Write

$$P = \begin{pmatrix} 0 & 0 \\ 0 & P_4 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 & * \\ 0 & u_1 \end{pmatrix}, \qquad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathrm{Mat}_k,$$

where $P_4, u_1, A_4 \in \mathrm{Mat}_{k-1}$. Then $u^t P u = \left(\begin{smallmatrix} 0 & 0 \\ 0 & u_1^t P_4 u_1 \end{smallmatrix}\right) = A$. Therefore the first row and the first column of A are both zeros and by induction assumption $A_4 = P_4$. Hence $A = P$.

(2) Assume that $P_{1,i} \neq 0$ for some $i \neq 2k$. Then the first row of $u^t P u = A$ is

$$(0, \ldots, 0, P_{1,i}, P_{1,i}u_{i,i+1}, \ldots, P_{1,i}u_{i,k}).$$

Since A is generalized monomial,

$$u_{i,i+1} = \cdots = u_{i,k} = 0 \quad \text{and} \quad A_{1,i} = P_{1,i}.$$

Write

$$P = \begin{pmatrix} P_1 & * & P_2 \\ * & 0 & 0 \\ P_3 & 0 & P_4 \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 & * & u_2 \\ 0 & 1 & 0 \\ 0 & 0 & u_3 \end{pmatrix}, \qquad A = \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix},$$

where $P_1, u_1, A_1 \in \mathrm{Mat}_{i-1}$, $P_4, u_3, A_9 \in \mathrm{Mat}_{k-i}$. Then

$$u^t P u = \begin{pmatrix} T_1 & * & T_2 \\ * & * & * \\ T_3 & * & T_4 \end{pmatrix},$$

where $\left(\begin{smallmatrix} T_1 & T_2 \\ T_3 & T_4 \end{smallmatrix}\right) = v^t Q v$, with

$$v = \begin{pmatrix} u_1 & u_2 \\ & u_3 \end{pmatrix} \in \mathrm{U}_{k-1} \quad \text{and} \quad Q = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in \mathrm{Mat}_{k-1}.$$

By induction assumption

$$\begin{pmatrix} A_1 & A_3 \\ A_7 & A_9 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}.$$

Also the i-th row and the i-th column of A and P are the same, since $A_{1,i} = P_{1,i}$ and they are both skew symmetric and monomial. Therefore $A = P$.

(3) Assume that $P_{1,2k} \neq 0$. Write

$$P = \begin{pmatrix} & & P_{1,2k} \\ & P_1 & \\ -P_{1,2k} & & \end{pmatrix}, \qquad u = \begin{pmatrix} 1 & * & * \\ & u_1 & * \\ & & 1 \end{pmatrix},$$

where P_1 is a $2k-2 \times 2k-2$ skew symmetric, generalized monomial matrix and $u_1 \in \mathrm{U}_{2k-2}$. Note that

$$u^t P u = \begin{pmatrix} & & P_{1,2k} \\ & u_1^t P_1 u_1 & * \\ -P_{1,2k} & * & * \end{pmatrix} = A.$$

By comparing the entries of $A = u^t P u$ and induction assumption, we obtain $A = P$. $\square$

4. Main theorem

We start with recalling results in [BJ03], which states the relation between irreducible representations of $\mathrm{O}(2n, \mathcal{F})$ and their restrictions to $\mathrm{SO}(2n, \mathcal{F})$, $n \geqslant 1$. Let

$$s = \begin{pmatrix} \mathrm{I}_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \mathrm{I}_{n-1} \end{pmatrix} \in \mathrm{O}(2n, \mathcal{F}).$$

Then

$$\mathrm{O}(2n, \mathcal{F}) = \mathrm{SO}(2n, \mathcal{F}) \rtimes \{1, s\},$$

where s acts on $\mathrm{SO}(2n, \mathcal{F})$ by conjugation. For a representation (V_{π_0}, π_0) of $\mathrm{SO}(2n, \mathcal{F})$, define another representation $s\pi_0$ on the same vector space V_{π_0} with action given by $s\pi_0(g) = \pi_0(sgs^{-1})$.

Denote by $\hat{s}$ the nontrivial character of $\mathrm{O}(2n, \mathcal{F})$ given by

$$\begin{cases} \hat{s}(g) = 1, & \\ \hat{s}(sg) = -1, & \text{for } g \in \mathrm{SO}(2n, \mathcal{F}). \end{cases}$$

Let $G_0 = \mathrm{SO}(2n, \mathcal{F})$ and $G = \mathrm{O}(2n, \mathcal{F})$.

Lemma 4.1. *(See Lemma 4.1 [BJ03].)*

(1) *For any admissible representation π_0 of G_0 and any admissible representation π of G_0 (π_0 and π are not necessarily irreducible), we have*

$$r^G_{G_0} \circ i^G_{G_0}(\pi_0) \cong \pi_0 \oplus s\pi_0, \quad i^G_{G_0}(\pi_0) \cong i^G_{G_0}(s\pi_0),$$
$$i^G_{G_0} \circ r^G_{G_0}(\pi) \cong \pi \oplus \hat{s}\pi, \quad r^G_{G_0}(\hat{s}\pi) \cong r^G_{G_0}(\pi),$$

where i denotes the induction functor and r denotes the restriction.

(2) *Let σ be an irreducible admissible representation of G and σ_0 be an irreducible subquotient of $r^G_{G_0}\sigma$. Then*

$$\sigma_0 \cong s\sigma_0 \quad \textit{if and only if} \quad \sigma \ncong \hat{s}\sigma.$$

(a) *If* $\sigma_0 \cong s\sigma_0$, *then*

$$i^G_{G_0}(\sigma_0) \cong \sigma \oplus \hat{s}\sigma \quad \text{and} \quad r^G_{G_0}(\sigma) \cong \sigma_0.$$

(b) *If* $\sigma_0 \ncong s\sigma_0$, *then*

$$i^G_{G_0}(\sigma_0) \cong \sigma \quad \text{and} \quad r^G_{G_0}(\sigma) \cong \sigma_0 \oplus s\sigma_0.$$

Lemma 4.2. [2] *Let* (V_π, π) *be an irreducible representation of* $\mathrm{SO}(2n, \mathcal{F})$. *Then its contragradient* $\tilde{\pi}$ *is isomorphic to* $s\pi$.

Proof. Denote by (V_σ, σ) an irreducible representation of $\mathrm{O}(2n, \mathcal{F})$. By the theorem (on p. 91, Sections II, II.I of [MVW87]), the contragradient $\tilde{\sigma}$ of σ is isomorphic to σ^δ, with σ^δ acting on V_σ by

$$\sigma^\delta(g) = \delta\sigma(g)\delta^{-1},$$

where δ satisfies $\delta^t \nu_{2n}\delta = \nu^t_{2n}$. By choosing δ to be $\nu_{2n} \in \mathrm{O}(2n, \mathcal{F})$, every irreducible representation of $\mathrm{O}(2n, \mathcal{F})$ is self-contragradient. Now Lemma 4.1 implies that $\tilde{\pi}$ is isomorphic to $s\pi$. □

Lemma 4.3. *If* π_0 *admits a nontrivial generalized Shalika model, then so does its contragradient* $\tilde{\pi}_0$.

Proof. Assume that

$$\pi_0 \ncong \tilde{\pi}_0 \quad \text{and note that} \quad \tilde{\pi}_0 \cong s\pi_0.$$

Write $\mathrm{Ind}^{\mathrm{O}(4n,\mathcal{F})}_{\mathrm{SO}(4n,\mathcal{F})}\pi_0 = \pi$. Since

$$\begin{aligned} &\dim \mathrm{Hom}_{\mathrm{O}(4n,\mathcal{F})}\big(\pi, \mathrm{Ind}^{\mathrm{O}(4n,\mathcal{F})}_{\mathcal{H}}\psi_{\mathcal{H}} = \mathrm{Ind}^{\mathrm{O}(4n)}_{\mathrm{SO}_{4n}}\big(\mathrm{Ind}^{\mathrm{SO}_{4n}}_{\mathcal{H}}\psi_{\mathcal{H}}\big)\big) \\ &\quad \cong \dim \mathrm{Hom}_{\mathrm{SO}(4n,\mathcal{F})}\big(r^{\mathrm{O}(4n,\mathcal{F})}_{\mathrm{SO}(4n,\mathcal{F})}\pi, \mathrm{Ind}^{\mathrm{SO}(4n,\mathcal{F})}_{\mathcal{H}}\psi_{\mathcal{H}}\big) \geqslant 1, \end{aligned}$$

we can find a function $0 \neq f \in \mathrm{Ind}^{\mathrm{O}(4n,\mathcal{F})}_{\mathcal{H}}\psi_{\mathcal{H}}$ such that

$$\begin{aligned} &\big(\big\langle \rho_g \cdot f \bigm| g \in \mathrm{SO}(4n, \mathcal{F})\big\rangle, \rho\big) \cong V_{\pi_0} \quad \text{and} \\ &\big(\big\langle \rho_g(\rho_s \cdot f) \bigm| g \in \mathrm{SO}(4n, \mathcal{F})\big\rangle, \rho\big) \cong V_{s\pi_0}, \end{aligned}$$

where ρ denotes the right translation. Denote

$$\tilde{f} = f|_{\mathrm{O}(4n,\mathcal{F})-\mathrm{SO}(4n,\mathcal{F})}.$$

Since f is an element in $\mathrm{Ind}^{\mathrm{O}(4n,\mathcal{F})}_{\mathcal{H}}\psi_{\mathcal{H}}$,

$$\hat{f} := \rho_s \cdot \tilde{f} : \mathrm{SO}(4n, \mathcal{F}) \mapsto \mathbb{C} \quad \text{satisfies } \hat{f}(hx) = \psi_{\mathcal{H}}(h)\hat{f}(x) \quad \text{for } h \in \mathcal{H},\ x \in \mathrm{SO}(4n, \mathcal{F}).$$

[2] I am very thankful to Dubravka Ban for her sharing this result and reference.

Therefore

$$\big(\big\langle \rho_g(\hat{f}) \bigm| g \in \mathrm{SO}(4n, \mathcal{F})\big\rangle, \rho\big) \cong V_{s\pi_0}$$

serves as a nontrivial generalized Shalika model for $V_{s\pi_0}$, which completes the proof. □

Lemma 4.4. *For each $g \in \mathrm{SO}_{4n}$, there exist $h, r \in \mathcal{H}_{2n}$ (depending on g) such that one of the following conditions holds*:

Condition 4. $hgr^{-1} = g$, and $\psi_{\mathcal{H}_{2n}}(hr^{-1}) \neq 1$.
Condition 5. $hgr^{-1} = g^{\tau}$, and $\psi_{\mathcal{H}_{2n}}(hr^{-1}) = 1$.

Proof. For $g \in \mathrm{SO}_{4n}(F)$, if

$$hgr^{-1} = g', \quad \text{such that} \quad \psi_{\mathcal{H}_{2n}}\big(hr^{-1}\big) = 1.$$

Then

$$(\epsilon h)g(\epsilon r)^{-1} = \epsilon g' \epsilon^{-1} = g^{\tau} \quad \text{and} \quad \psi_{\mathcal{H}_{2n}}\big((\epsilon h)(\epsilon r)^{-1}\big) = 1. \qquad \square$$

Now we ready for the proof of our main theorem.

Proof of Theorem 1.9. Let F be either a finite field with characteristic p, $p \neq 2$, or a p-adic field.

(1) In the case of finite fields, we obtain

$$\mathrm{Hom}_{\mathrm{SO}(4n,\mathbb{F}_q)}\big(\mathrm{Ind}_{\mathcal{H}}^{\mathrm{SO}(4n,\mathbb{F}_q)} \psi_{\mathcal{H}}, \mathrm{Ind}_{\mathcal{H}}^{\mathrm{SO}(4n,\mathbb{F}_q)} \psi_{\mathcal{H}}\big) \quad \text{is abelian,}$$

which implies the uniqueness of the generalized Shalika model for $\mathrm{SO}(4n, \mathbb{F}_q)$ as we mentioned in Section 2.

(2) In the case of p-adic fields, by Theorem 2.2, we obtain

$$\dim \mathrm{Hom}_{\mathrm{SO}(4n,\mathcal{F})}\big(\sigma_0; \mathrm{Ind}_{\mathcal{H}}^{\mathrm{SO}(4n,\mathcal{F})} \psi_{\mathcal{H}}\big) \cdot \dim \mathrm{Hom}_{\mathcal{H}}\big(\mathrm{Res}_{\mathcal{H}}^{\mathrm{SO}(4n,\mathcal{F})} \tilde{\sigma}_0; \psi_{\mathcal{H}}\big) \leqslant 1,$$

for any irreducible representation σ_0 of $\mathrm{SO}(4n, \mathcal{F})$. By reciprocity,

$$\mathrm{Hom}_{\mathcal{H}}\big(\mathrm{Res}_{\mathcal{H}}^{\mathrm{SO}(4n,\mathcal{F})} \tilde{\sigma}_0; \psi_{\mathcal{H}}\big) \cong \mathrm{Hom}_{\mathrm{SO}_{4n}}\big(\tilde{\sigma}_0; \mathrm{Ind}_{\mathcal{H}}^{\mathrm{SO}(4n,\mathcal{F})} \psi_{\mathcal{H}}\big).$$

Assume that σ_0 admits a nontrivial generalized Shalika model, and then so does its contragradient $\tilde{\sigma}_0$ by Lemma 4.3. Hence $\dim \mathrm{Hom}_{\mathrm{SO}(4n,\mathcal{F})}(\sigma_0; \mathrm{Ind}_{\mathcal{H}}^{\mathrm{SO}(4n,\mathcal{F})} \psi_{\mathcal{H}}) \leqslant 1$ and the uniqueness of generalized Shalika model holds. □

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